On p-Adic Vector Measure Spaces

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Abstract

For $\mathcal{R}$ be a separating algebra of subsets of a set X, E a complete Hausdorff non-Archimedean locally convex space and $m : \mathcal{R} \to E$ a bounded finitely additive measure, we study some of the properties of the integrals with respect to m of scalar valued functions on X. The concepts of convergence in measure, with respect to m, and of m-measurable functions are introduced and several results concerning these notions are given.

1 Preliminaries

Throughout this paper, $\mathbb{K}$ will be a complete non-Archimedean valued field, whose valuation is non-trivial. By a seminorm, on a vector space over $\mathbb{K}$, we will mean a non-Archimedean seminorm. Similarly, by a locally convex space we will mean a non-Archimedean locally convex space over $\mathbb{K}$ (see [10] or [11]). For E a locally convex space, we will denote by cs(E) the collection of all continuous seminorms on E. For X a set, $f \in \mathbb{K}^X$ and $A \subset X$, we define

$$\|f\|_A = \sup\{|f(x)| : x \in A\} \quad \text{and} \quad \|f\| = \|f\|_X.$$

Also for $A \subset X$, A^c will be its complement in X and χ_A the $\mathbb{K}$-valued characteristic function of A. The family of all subsets of X will be denoted by $P(X)$.

Assume next that X is a non-empty set and $\mathcal{R}$ a separating algebra of subsets of X, i.e. $\mathcal{R}$ is a family of subsets of X such that

1. $X \in \mathcal{R}$, and, if $A, B \in \mathcal{R}$, then $A \cup B$, $A \cap B$, A^c are also in $\mathcal{R}$.

2. If x, y are distinct elements of X, then there exists a member of $\mathcal{R}$ which contains x but not y.

Then $\mathcal{R}$ is a base for a Hausdorff zero-dimensional topology $\tau_{\mathcal{R}}$ on X. For E a locally convex space, we denote by $M(\mathcal{R}, E)$ the space of all finitely-additive measures $m : \mathcal{R} \to E$ such that $m(\mathcal{R})$ is a bounded subset of E (see [7]). For a net (V_δ) of subsets of X, we write $V_\delta \downarrow \emptyset$ if (V_δ) is decreasing and $\cap V_\delta = \emptyset$. An element $m \in M(\mathcal{R}, E)$ is said to be σ-additive if $m(V_n) \to 0$ for each sequence (V_n) in $\mathcal{R}$ which decreases to the empty set. We denote by $M_\sigma(\mathcal{R}, E)$ the space of all σ-additive members of $M(\mathcal{R}, E)$. An m of $M(\mathcal{R}, E)$ is said to be τ-additive if $m(V_\delta) \to 0$ for each net (V_δ) in $\mathcal{R}$ with $V_\delta \downarrow \emptyset$. We will denote by $M_\tau(\mathcal{R}, E)$ the space of all τ-additive members of $M(\mathcal{R}, E)$. For $m \in M(\mathcal{R}, E)$ and $p \in cs(E)$, we define

$$m_p : \mathcal{R} \to \mathbb{R}, \quad m_p(A) = \sup\{p(m(V)) : V \in \mathcal{R}, V \subset A\} \quad \text{and} \quad \|m\|_p = m_p(X).$$

We also define

$$N_{m,p} : X \to \mathbb{R}, \quad N_{m,p}(x) = \inf\{m_p(V) : x \in V \in \mathcal{R}\}.$$

Next we will recall the definition of the integral of an $f \in \mathbb{K}^X$ with respect to some $m \in M(\mathcal{R}, E)$. Assume that E is a complete Hausdorff locally convex space. For $A \subset X$, let $\mathcal{D}_A$ be the family of all $\alpha = \{A_1, A_2, \dots, A_n; x_1, x_2, \dots, x_n\}$, where $\{A_1, A_2, \dots, A_n\}$ is an $\mathcal{R}$-partition of A and $x_k \in A_k$. We make $\mathcal{D}_A$ into a directed set by defining $\alpha_1 \geq \alpha_2$ if the partition of A in α_1 is a refinement of the one in α_2. For $\alpha = \{A_1, A_2, \dots, A_n; x_1, x_2, \dots, x_n\}$, we define $\omega_\alpha(f, m) = \sum_{k=1}^n f(x_k)m(A_k)$. If the limit $\lim \omega_\alpha(f, m)$ exists in E, we will say that f is m-integrable over A and denote this limit by $\int_A f\, dm$. For $A = X$, we write simply $\int f\, dm$. It is easy to see that if f is m-integrable over X, then it is m-integrable over every $A \in \mathcal{R}$ and $\int_A f\, dm = \int \chi_A f\, dm$. If f is bounded on A, then

$$p\left(\int_A f\, dm\right) \leq \|f\|_A \cdot m_p(A).$$

2 Measurable Sets

Throughout the paper, $\mathcal{R}$ will be a separating algebra of subsets of a set X, E a complete Hausdorff locally convex space and $M(\mathcal{R}, E)$ the space of all bounded E-valued finitely-additive measures on $\mathcal{R}$. We will denote by $\tau_{\mathcal{R}}$ the topology on X which has $\mathcal{R}$ as a basis. Every member of $\mathcal{R}$ is $\tau_{\mathcal{R}}$-clopen, i.e both closed and open. By $S(\mathcal{R})$ we will denote the space of all $\mathbb{K}$-valued $\mathcal{R}$-simple functions. As in [7], if $m \in M(\mathcal{R}, E)$, then a subset A of X is said to be m-measurable if the characteristic function χ_A is m-integrable. By [7, Theorem 4.7], A is measurable iff, for each $p \in cs(E)$ and each $\epsilon > 0$, there

exist V, W in $\mathcal{R}$ such that $V \subset A \subset W$ and $m_p(W \setminus V) < \epsilon$.
Let $\mathcal{R}_m$ be the family of all m-measurable sets. By [7] we have the following

Theorem 2.1 *1. $\mathcal{R}_m$ is an algebra of subsets of X.*

2. If $\bar{m} : \mathcal{R}_m \to E$, $\bar{m}(A) = \int \chi_A \, dm$, then $\bar{m} \in M(\mathcal{R}_m, E)$.

3. $\bar{m}$ is σ-additive iff m is σ-additive.

4. $\bar{m}$ is τ-additive iff m is τ-additive.

5. For $p \in cs(E)$, we have $N_{m,p} = N_{\bar{m},p}$.

6. $\mathcal{R}_m = \mathcal{R}_{\bar{m}}$.

7. For $A \in \mathcal{R}$, we have $m_p(A) = \bar{m}_p(A)$.

8. For $A \in \mathcal{R}_m$, we have

$$\bar{m}_p(A) = \inf\{m_p(W) : W \in \mathcal{R}, A \subset W\}.$$

9. If $f \in \mathbb{K}^X$ is m-integrable, then f is $\bar{m}$-integrable and $\int f \, dm = \int f \, d\bar{m}$.

10. If f is bounded and $\bar{m}$-integrable, then f is m-integrable.

11. An $f \in \mathbb{K}^X$ is m-integrable iff, for each $p \in cs(E)$ and each $\epsilon > 0$, there exists an $\mathcal{R}$-partition $\{A_1, \ldots, A_n\}$ of X such that, for each $1 \le k \le n$, we have $|f(x) - f(y)| \cdot m_p(A_k) \le \epsilon$ if $x, y \in A_k$. In this case, if $x_k \in A_k$, then

$$p\left(\int f \, dm - \sum_{k=1}^{n} f(x_k) m(A_k)\right) \le \epsilon.$$

12. If m is τ-additive, then a subset A of X is measurable iff A is $\tau_{\mathcal{R}_m}$-clopen.

For $m \in M(\mathcal{R}, E)$ and $p \in cs(E)$, we define

$$m_p^* : P(X) \to \mathbb{R}, \quad m_p^*(A) = \inf\{m_p(W) : A \subset W \in \mathcal{R}\}.$$

It is easy to see that

$$m_p^*(A \cup B) = \max\{m_p^*(A), m_p^*(B)\}.$$

By [7, Theorem 4.10], we have that $m_p^*(A) = \bar{m}_p(A)$ for all $A \in \mathcal{R}_m$.
For $p \in cs(E)$, define

$$d_p : P(X) \times P(X) \to \mathbb{R}, \quad d_p(A, B) = m_p^*(A \triangle B),$$

where $A\triangle B = (A\backslash B)\bigcup(B\backslash A)$. It is easy to see that d_p is an ultrapseudometric on $P(X)$. Let $\mathcal{U}_m$ be the uniformity induced by the family of pseudometrics d_p, $p \in cs(E)$.
For A, B in $\mathcal{R}$, we have

$$p(m(A) - m(B)) \leq m_p(A\triangle B) = m_p(A, B).$$

Hence $m : \mathcal{R} \to E$ is $\mathcal{U}_m$-uniformly continuous. Let G_m be the closure of $\mathcal{R}$ in $(P(X), \mathcal{U}_m)$. Then m has a unique uniformly continuous extension $\hat{m} : G_m \to E$.

Theorem 2.2 *$G_m = \mathcal{R}_m$ and $\hat{m} = \bar{m}$.*

Proof : Assume that $A \in G_m$ and let $p \in cs(E)$ $\epsilon > 0$. There exists $V_1 \in \mathcal{R}$ such that $m_p^*(A\triangle V_1) < \epsilon$. Let W_1 in $\mathcal{R}$ be such that $A\triangle V_1 \subset W_1$ and $m_p(W_1) < \epsilon$. Let $V = V_1 \cap W_1^c$, $W = V_1 \cup W_1$. Then $V \subset A \subset W$. Moreover, $W\backslash V = W_1$, and so $m_p(W \setminus V) < \epsilon$, which proves that $A \in \mathcal{R}_m$. Conversely, suppose that $A \in \mathcal{R}_m$ and let V, W in $\mathcal{R}$ be such that $V \subset A \subset W$ and $m_p(W \setminus V) < \epsilon$. Since $A\triangle V = A \setminus V \subset W \setminus V$, we have that $m_p^*(A\triangle V) \leq m_p(W \setminus V) < \epsilon$, which proves that $A \in G_m$. Finally, for A, B in $\mathcal{R}_m$, we have

$$p(\bar{m}(A) - \bar{m}(B)) = p(\bar{m}(A\triangle B)) \leq \bar{m}_p(A\triangle B) = d_p(A, B).$$

Hence $\bar{m}$ is a $\mathcal{U}_m$-uniformly continuous extension of m and so $\bar{m} = \hat{m}$. This completes the proof.

Definition 2.3 *If $m \in M(\mathcal{R}, E)$, then a subset A of X is said to be m-negligible if $m_p^*(A) = 0$ for every $p \in cs(E)$. A property concerning elements of X is said to be true almost everywhere with respect to m (in short m-a.e) if the set of all points in X for which it is false is m-negligible.*

It is clear that every m-negligible set is measurable.

Theorem 2.4 *Let $m \in M_\sigma(\mathcal{R}, E)$ and suppose that $\mathcal{R}$ is a σ-algebra. Then :*

1. *A subset B of X is measurable iff, for each $p \in cs(E)$, there are $V, W \in \mathcal{R}$ with $V \subset B \subset W$ and $m_p(V) = m_p(W) = m_p^*(B)$, $m_p(W \setminus V) = 0$.*

2. *$\mathcal{R}_m$ is a σ-algebra.*

3. *If E is metrizable, then B is measurable iff there are a $V \in \mathcal{R}$ and an m-negligible set A such that $B = A \cup V$.*

Proof : 1. Suppose that B s measurable. There are an increasing sequence (V_n) in $\mathcal{R}$ and a decreasing sequence (W_n) in $\mathcal{R}$ such that $V_n \subset B \subset W_n$

and $m_p(W_n \setminus V_n) < 1/n$. Let $V = \bigcup V_n$, $W = \bigcap W_n$. Then $V, W \in \mathcal{R}$ and $m_p(W \setminus V) = 0$. Since $B = V \bigcup (B \setminus V) \subset V \bigcup (W \setminus V)$, we have that

$$m_p^*(B) = \bar{m}_p(B) \leq \max\{m_p(V), m_p(W \setminus V)\} = m_p(V) \leq m_p^*(B)$$

and so $m_p^*(B) = m_p(V)$. Analogously we prove that $m_p(W) = m_p^*(B)$.
2. Let (A_n) be a sequence in $\mathcal{R}_m$, $A = \bigcup A_n$, $p \in cs(E)$ and $\epsilon > 0$. For each n, there are $V_n, W_n \in \mathcal{R}$ with $V_n \subset A_n \subset W_n$ and $m_p(W_n \setminus V_n) < \epsilon$. The sets $V = \bigcup V_n$, $W = \bigcup W_n$ are in $\mathcal{R}$ and $W \setminus V \subset \bigcup_{n=1}^{\infty} W_n \setminus V_n$, and therefore $m_p(W \setminus V) \leq \sup_n m_p(W_n \setminus V_n) \leq \epsilon$. This proves that $A \in \mathcal{R}_m$.
3. Suppose that E is metrizable and let (p_n) be an increasing sequence of continuous seminorms on E such that, for each $p \in cs(E)$, there exists n with $p \leq p_n$. Assume that B is measurable. For each n, there are $V_n, W_n \in \mathcal{R}$ with $V_n \subset B \subset W_n$ and $m_{p_n}(W_n \setminus V_n) = 0$. Let $V = \bigcup V_n$, $W = \bigcap W_n$. Then $V, W \in \mathcal{R}$. Given $p \in cs(E)$, there exists n such that $p \leq p_n$ and so

$$m_p(W \setminus V) \leq m_{p_n}(W \setminus V) \leq m_{p_n}(W_n \setminus V_n) = 0.$$

The set $A = B \setminus V \subset W \setminus V$ is m-negligible and $B = V \cup A$. Hence the result follows.

Theorem 2.5 *Let $m \in M_\sigma \mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let (A_n) be a sequence of measurable subsets of Xwhich converges to some A in $P(X)$ with respect to the topology induced by the uniformity $\mathcal{U}_m$. Let*

$$B_1 = \liminf A_n = \bigcup_n \bigcap_{k \geq n} A_k, \quad B_2 = \limsup A_n = \bigcap_n \bigcup_{k \geq n} A_k.$$

Then A is measurable and the sets $B_2 \setminus B_1$, $A \triangle B_1$ and $A \triangle B_2$ are m-negligible. Moreover $A_n \to B_1$ and $A_n \to B_2$.

Proof : Since $\mathcal{R}_m$ is closed in $P(X)$, it follows that A is measurable. Let $p \in cs(E)$ and $\epsilon > 0$. There exists n_o such that $\bar{m}_p(A \triangle A_n) < \epsilon$ for all $n \geq n_o$. Since

$$A \setminus B_2 \subset A \setminus B_1 = \bigcap_n \bigcup_{k \geq n} A \setminus A_k,$$

we have that

$$\bar{m}_p(A \setminus B_2) \leq \bar{m}_p(A \setminus B_1) \leq \bar{m}_p\left(\bigcup_{k \geq n_o} (A \setminus A_k)\right) = \sup_{k \geq n_o} \bar{m}_p(A \setminus A_k) \leq \epsilon.$$

Also

$$B_1 \setminus A \subset B_2 \setminus A = \bigcap_n \left(\bigcup_{k \geq n} A_k \setminus A\right) \subset \bigcup_{k \geq n_o} (A_k \setminus A)$$

and so $\bar{m}_p(B_1 \setminus A) \leq \bar{m}_p(B_2 \setminus A) \leq \epsilon$. This, being true for each $\epsilon > 0$, implies that the sets $B_1 \triangle A$ and $B_2 \triangle A$ are m-negligible. Moreover $B_1 \triangle B_2 \subset (B_1 \triangle A) \bigcup (B_2 \triangle A)$, and so $B_1 \triangle B_2$ is m-negligible. Finally,

$$A_n \triangle B_1 \subset (A_n \triangle A) \cup (A \triangle B_1)$$

and so $\bar{m}_p(A_n \triangle B_1 \leq \bar{m}_p(A_n \triangle A) \to 0$, which proves that $A_n \to B_1$. Similarly $A_n \to B_2$.

Theorem 2.6 *Let $m \in M_\sigma(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let $f \in \mathbb{K}^X$. Then, f ia m-integrable iff it is $\bar{m}$-integrable. Moreover*

$$\int f\, dm = \int f\, d\bar{m}.$$

Proof : By Theorem 2.1, if f is m-integrable, then it is also $\bar{m}$-integrable and the two integrals coincide. Conversely, suppose that f is $\bar{m}$-integrable and let $p \in cs(E)$ and $\epsilon > 0$. By Theorem 2.1, there exists an $\mathcal{R}_m$-partition $\{A_1, \ldots, A_n\}$ of X such that, for each $k = 1, 2, \ldots$, we have $|f(x) - f(y)| \cdot \bar{m}_p(A_k) < \epsilon$ if $x, y \in A_k$. In view of Theorem 2.4, there are sets $V_k, W_k \in \mathcal{R}$ with $V_k \subset A_k \subset W_k$ and $m_p(W_k \setminus V_k) = 0$, $m_p(V_k) = \bar{m}_p(A_k)$. Let $V_{n+1} = X \setminus \bigcup_{k=1}^n V_k$. Then $V_{n+1} \subset \bigcup_{k=1}^n W_k \setminus V_k$ and so $m_p(V_{n+1}) = 0$. Now $\{V_1, V_2, \ldots, V_{n+1}\}$ is an $\mathcal{R}$-partition of X and, for $0 \leq k \leq n+1$, we have $|f(x) - f(y)| \cdot m_p(V_k) < \epsilon$, if $x, y \in A_k$, which proves that f is m-integrable by Theorem 2.1.

Definition 2.7 *Let $m \in M(\mathcal{R}, E)$ and $f \in \mathbb{K}^X$. We say that f is m-integrable over a measurable set A if $f \cdot \chi_A$ is m-integrable over X. In this case we define*

$$\int_A f\, dm = \int f \chi_A\, dm.$$

If f is m-integrable, then f is $\bar{m}$-integrable. Also χ_A is $\bar{m}$-integrable and so $f\chi_A$ is $\bar{m}$-integrable over X (by [7, Theorem 4.3), which implies that $f\chi_A$ is m-integable. Moreover

$$\int_A f\, dm = \int f\, \chi_A dm = \int f\chi_A\, d\bar{m} = \int_A f\, d\bar{m}.$$

Theorem 2.8 *Let $m \in M(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$ be m-integrable. Then, given $\epsilon > 0$, there exists $\delta > 0$ such that $p\left(\int_A f\, dm\right) < \epsilon$ for each $A \in \mathcal{R}_m$ with $\bar{m}_p(A) < \delta$.*

Proof : Since f is m-integrable, there exists $W \in \mathcal{R}$ such that $m_p(W^c) = 0$ and $\|f\|_W < d < \infty$. Let $\delta = \epsilon/d$ and let $A \in \mathcal{R}_m$ with $\bar{m}_p(A) < \delta$. Then

$$p\left(\int_A f\, dm\right) = p\left(\int_A f\, d\bar{m}\right) = p\left(\int_{A \cap W} f\, d\bar{m}\right) \leq \|f\|_{A \cap W} \cdot \bar{m}_p(A \cap W) < \epsilon.$$

Theorem 2.9 *Let $m \in M_\tau(\mathcal{R}, E)$and let $f \in \mathbb{K}^X$. Then f is m-integrable iff*

1. *f is $\tau_{\mathcal{R}}$-continuous at every point of the set*

$$G = \bigcup_{p \in cs(E)} \{x : N_{m,p}(x) \neq 0\}.$$

2. *For every $p \in cs(E)$, there exists $W \in \mathcal{R}$, with $m_p(W^c) = 0$ and $\|f\|_W < \infty$.*

Proof : The necessity follows from [7, Theorem 4.2].
Conversely, suppose that (1) and (2) hold and let $p \in cs(E)$ and $\epsilon > 0$. Let $W \in \mathcal{R}$ be such that $m_p(W^c) = 0$ and $\|f\|_W < d < \infty$. Let $\epsilon_1 > 0$ be such that $\epsilon_1 d < \epsilon$ and $\epsilon_1 \cdot \|m\|_p < \epsilon$. The set $Y = \{x : N_{m,p}(x) \geq \epsilon_1\}$ is $\tau_{\mathcal{R}}$-compact (by [7, Theorem 2.6]) and it is contained in W. By (2), f is $\tau_{\mathcal{R}}$- continuous at every point of Y. Hence, for each $x \in Y$, there exists V_x in $\mathcal{R}$ contained in W such that

$$x \in V_x \subset \{y : |f(y) - f(x)| < \epsilon_1\}.$$

By the compactness of Y, Y is covered by a finite number of the V_x, $x \in Y$. Thus, there are pairwise disjoint members $A_1, A_2, \ldots, A_n$ of $\mathcal{R}$ which cover Y such that $A_k \subset W$ and each A_k is contained in some V_x. Let $A_{n+1} = W \setminus \bigcup_1^n A_k$, $A_{n+2} = W^c$. Then

$$m_p(A_{n+1}) = \sup_{x \in A_{n+1}} N_{m,p}(x) \leq \epsilon_1$$

(by [7, Corollary 2.3]) and so

$$|f(x) - f(y)| \cdot m_p(A_{n+1}) \leq d\epsilon_1 < \epsilon$$

if $x, y \in A_{n+1}$. If $x, y \in A_k$, for some $k \leq n$, then

$$|f(x) - f(y)| \cdot m_p(A_k) \leq \epsilon_1 \cdot m_p(A_k) < \epsilon.$$

Now the result follows by Theorem 2.1.

Theorem 2.10 *If $f = g$ m-a.e and g is m-integrable, then f is m-integrable and*

$$\int f\, dm = \int g\, dm.$$

Proof : We will show that f is $\bar{m}$-integrable . The set $A = \{x : f(x) \neq g(x)\}$ is m-negligible and hence $A \in \mathcal{R}_m$. Since g is m-integrable, given $\epsilon > 0$ and $p \in cs(E)$, there exists an $\mathcal{R}$-partition $\{A_1, A_2, \ldots, A_n\}$ of X such that $|g(x) - g(y)| \cdot m_p(A_k) < \epsilon$ if $x, y \in A_k$. If now $\{B_1, B_2, \ldots, B_N\}$ is any $\mathcal{R}_m$-partition of X which is a refinement of each of the partitions $\{A_1, A_2, \ldots, A_n\}$

and $\{A, A^c\}$, then $|f(x) - f(y)| \cdot \bar{m}_p(B_k) < \epsilon$ if $x, y \in B_k$. Indeed this clearly holds if $B_k \subset A$. If $B_k \subset A^c$, then

$$|f(x) - f(y)| \cdot \bar{m}_p(B_k) = |g(x) - g(y)| \cdot \bar{m}_p(B_k) < \epsilon$$

since each B_k is contained in some A_j. This (in view of Theorem 2.1) implies that f is $\bar{m}$-integrable and hence m-integrable. By the same Theorem, if $x_k \in B_k$, then

$$p\left(\int f\, d\bar{m} - \sum_{k=1}^{N} f(x_k)\bar{m}(B_k)\right) \leq \epsilon \quad \text{and} \quad p\left(\int g\, d\bar{m} - \sum_{k=1}^{N} g(x_k)\bar{m}(B_k)\right) \leq \epsilon.$$

Since, for $B_k \subset A$, we have that $\bar{m}(B_k = 0$ and $f(x_k) = g(x_k)$ when $B_k \subset A^c$, it follows that

$$p\left(\int f\, d\bar{m} - \int g\, d\bar{m}\right) \leq \epsilon.$$

This, being true for all $\epsilon > 0$ and all $p \in cs(E)$, implies that

$$\int f\, dm = \int f\, d\bar{m} = \int g\, d\bar{m} = \int g\, dm,$$

which completes the proof.

Theorem 2.11 *Let $m \in M_\sigma(\mathcal{R}, E)$ and suppose that $\mathcal{R}$ is a σ-algebra. If (A_n) is a sequence in $\mathcal{R}$, then for each $p \in cs(E)$ we have*

$$m_p(\liminf A_n) \leq \liminf m_p(A_n) \leq \limsup m_p(A_n) \leq m_p(\limsup A_n).$$

Proof : Let $B_n = \bigcap_{k=n}^{\infty} A_k, \quad G_n = \bigcup_{k=n}^{\infty} A_k$. Then

$$\liminf A_n = \bigcup B_n \quad \text{and} \quad \limsup A_n = \bigcap G_n.$$

Since m is σ-additive, we have $m_p(\liminf A_n) = \sup_n m_p(B_n)$. But

$$m_p(B_n) \leq \inf_{k \geq n} m_p(A_k) \leq \liminf m_p(A_n).$$

Thus

$$m_p(\liminf A_n) \leq \liminf m_p(A_n).$$

Analogously we prove that

$$\limsup m_p(A_n) \leq m_p(\limsup A_n)$$

and hence the result follows.

Corollary 2.12 *Let $m \in M_\sigma(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let (A_n) be a sequence in $\mathcal{R}$ such that*

$$\liminf A_n = \limsup A_n = A.$$

Then, for each $p \in cs(E)$, we have that $m_p(A_n) \to m_p(A)$.

Theorem 2.13 *Let $m \in M(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$ be m-integrable. If $p \in cs(E)$ $\alpha > 0$ ad $\epsilon > 0$, then there exists $g \in S(\mathcal{R})$ such that*

$$m_p^*(\{x : |f(x) - g(x)| \geq \alpha\}) \leq \epsilon.$$

Proof : Since f is m-integrable, there exists an $\mathcal{R}$-partition $\{A_1, A_2, \ldots, A_n\}$ of X such that $|f(x) - f(y)| \cdot m_p(A_k) \leq \epsilon\alpha$ if $x, y \in A_k$. Let $x_k \in A_k$, $g = \sum_{k=1}^n f(x_k)\chi_{A_k}$ and $G = \{x : |f(x) - g(x) \geq \alpha\}$. If $x \in G \cap A_k$, then

$$\epsilon\alpha \geq |f(x) - f(x_k)| \cdot m_p(A_k) \geq \alpha \cdot m_p(A_k)$$

and thus $m_p(A_k) \leq \epsilon$. The set

$$W = \bigcup\{A_k : A_k \cap G \neq \emptyset\}$$

contains G and so $m_p^*(G) \leq m_p(W) \leq \epsilon$.

Theorem 2.14 *Let $m \in M(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$ be m-integrable. Then, for each $\alpha > 0$, the sets*

$$A_1 = \{x : |f(x)| \geq \alpha\}, \quad A_2 = \{x : |f(x)| > \alpha\}, \quad A_3 = \{x : |f(x)| \leq \alpha\}$$

$$A_4 = \{x : |f(x)| < \alpha\} \quad and \quad A_5 = \{x : |f(x)| = \alpha\}$$

are m-measurable.

Proof : Let $p \in cs(E)$ and $\epsilon > 0$. By the preceding Theorem, there exists $W \in \mathcal{R}$ and $g \in S(\mathcal{R})$ such that $m_p(W) < \epsilon$ and $\{x : |f(x) - g(x)| \geq \alpha\} \subset W$. Let $g = \sum_{k=1}^n \lambda_k \chi_{B_k}$, where $B_1, \ldots, B_n$ are disjoint members of $\mathcal{R}$. Let $B = \{B_k : |\lambda_k| \geq \alpha\}$. Then

$$B \cap W^c \subset \{x : |f(x)| \geq \alpha\} \subset W \cup B.$$

Indeed, let $x \in B \cap W^c$ and assume that $|f(x)| < \alpha$. Since $x \in B$, we have $|g(x)| \geq \alpha$ and so $|g(x) - f(x)| = |g(x)| \geq \alpha$, a contradiction. Hence $B \cap W^c \subset A_1$. Also, if $y \notin W \cup B$, then $|f(y) - g(y)| < \alpha$ and $|g(y)| < \alpha$, which implies that $|f(y)| < \alpha$. Thus $A_1 \subset B \cup W$. Moreover $(W \cup B) \setminus (B \cap W^c) = W$ and $m_p(W) < \epsilon$. This proves that A_1 is m-measurable. In an analogous way we prove that A_2 is measurable. Finally the sets $A_3 = A_2^c$, $A_4 = A_1^c$, and $A_5 = A_1 \setminus A_2$ are measurable.

3 Measurable Functions

Definition 3.1 *If $m \in M(\mathcal{R}, E)$, then a function $f \in \mathbb{K}^X$ is said to be m-measurable, or just measurable if no confusion is possible to arise, if $f^{-1}(A) \in \mathcal{R}_m$ for each clopen subset A of $\mathbb{K}$.*

We have the following two easily verified Lemmas.

Lemma 3.2 *A subset A of X is measurable iff χ_A is measurable.*

Lemma 3.3 *Let A be a closed subset of $\mathbb{K}$ and let*

$$\omega_A : \mathbb{K} \to \mathbb{R}, \quad \omega_A(x) = \inf_{y \in A} |x - y|.$$

Then :

1. *For $x, y \in \mathbb{K}$, we have $\omega_A(x) \leq \max\{|x - y|, \omega_A(y)\}$.*
2. *For each $\alpha > 0$, the sets*

   $$\{x : \omega_A(x) \leq \alpha\}, \quad \{x : \omega_A(x) < \alpha\} \quad \{x : \omega_A(x) \geq \alpha\}, \quad \{x : \omega_A(x) > \alpha\}$$

   are clopen.

Theorem 3.4 *Let $m \in M(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let $f \in \mathbb{K}^X$. The following are equivalent :*

1. *For each Borel subset B of $\mathbb{K}$, the set $f^{-1}(B)$ is measurable.*
2. *$f^{-1}(A)$ is measurable for each closed subset A of $\mathbb{K}$.*
3. *$f^{-1}(A)$ is measurable for each open subset A of $\mathbb{K}$.*
4. *f is measurable.*

Proof : It is clear that (2) is equivalent to (3) and that (1) $\Rightarrow$ (2) $\Rightarrow$ (4). Also, (3) $\Rightarrow$ (1) since the family of all subsets A of $\mathbb{K}$ for which $f^{-1}(A) \in \mathcal{R}_m$ is a σ-algebra because $\mathcal{R}_m$ is a σ-algebra. Finally, (4) implies (2). Indeed assume that f is measurable and let A be a closed subset of $\mathbb{K}$. Let ω_A be as in the preceding Lemma. Since A is closed, we have that $A = \{s \in \mathbb{K} : \omega_A(s) = 0\}$. Let $A_n = \{s : \omega_A(s) \leq 1/n\}$. Each A_n is clopen and thus $B_n = f^{-1}(A_n)$ is measurable. Since $f^{-1}(A) = \bigcap B_n$, the result clearly follows.

Theorem 3.5 *Let $m \in M(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$ be m-measurable. Then :*

1. *If $\phi : \mathbb{K} \to \mathbb{K}$ is continuous, then the function $\phi \circ f$ is measurable.*
2. *For each $g \in S(\mathcal{R}_m)$, the functions $h_1 = gf$ and $h_2 = g + f$ are measurable.*

Proof : 1). It follows from the fact that $\phi^{-1}(A)$ is clopen in $\mathbb{K}$ for each clopen A.
2). There exists an $\mathcal{R}_m$-partition $\{A_1, \ldots, A_n\}$ of X, and λ_k in $\mathbb{K}$ such that $g = \sum_{k=1}^{n} \lambda_k \chi_{A_k}$, $\lambda_n = 0$, $\lambda_k \neq 0$ for $k < n$ (we may have $A_n = \emptyset$). Now, for A clopen subset of $\mathbb{K}$, we have

$$h_1^{-1}(A) = \bigcup_{k=1}^{n} h_1^{-1}(A) \cap A_k.$$

If $k < n$, then

$$h_1^{-1}(A) \cap A_k = A_k \bigcap \left[f^{-1}(\lambda_k^{-1} A)\right].$$

Also

$$h_1^{-1}(A) \cap A_n \in \{A_n, \emptyset\}.$$

Hence each $h_1^{-1}(A) \cap A_k$ is measurable and so $h_1^{-1}(A)$ is measurable, which proves that h_1 is measurable. To prove that h_2 is measurable, it suffices to show that, for $G \in \mathcal{R}_m$ and $\lambda \in \mathbb{K}$, the function $h = f + \lambda \chi_G$ is measurable. For such an h and A clopen subset of $\mathbb{K}$, we have

$$h^{-1}(A) = \left[G \cap f^{-1}(-\lambda + A)\right] \bigcup \left[G^c \cap f^{-1}(A)\right],$$

and the result follows.

Theorem 3.6 *Let $m \in M_\tau(\mathcal{R}, E)$. Then :*

1. *An $f \in \mathbb{K}^X$ is measurable iff it is $\tau_{\mathcal{R}_m}$-continuous.*

2. *If f, g are measurable, then $f + g$ and fg are measurable.*

Proof : 1). It follows from the fact that, when m is τ-additive, a subset of X is in $\mathcal{R}_m$ iff it is $\tau_{\mathcal{R}_m}$-clopen.
2). It is a consequence of (1) since the sum and the product of two continuous functions are continuous.

Theorem 3.7 *Let $m \in M(\mathcal{R}, E)$ and let $f, g \in \mathbb{K}^X$ with $f = g$ m-a.e. If g is measurable, then f also is measurable.*

Proof : The set $G = \{x : f(x) \neq g(x)\}$ is negligible and hence measurable. For A a clopen subset of $\mathbb{K}$, we have

$$f^{-1}(A) = \left[f^{-1}(A) \cap G\right] \bigcup \left[f^{-1}(A) \cap G^c\right] = \left[f^{-1}(A) \cap G\right] \bigcup \left[g^{-1}(A) \cap G^c\right].$$

Since $f^{-1}(A) \cap G$ is negligible and hence measurable, the result follows.

Theorem 3.8 *Let $m \in M(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra. If f, g are measurable functions and $\lambda \in \mathbb{K}$, then the sets*

$$G_1 = \{x : |f(x)| > |g(x)|\}, \quad G_2 = \{x : |f(x)| \geq |g(x)|\},$$

$$G_3 = \{x : |f(x)| = |g(x)|\}, \quad G_4 = \{x : f(x) = \lambda\}$$

are measurable.

Proof : For each rational number r, the set

$$F_r = \{x : |f(x)| > r\} \bigcap \{x : x : |g(x)| < r\}$$

is measurable. Since $\mathcal{R}$ is a σ-algebra, $\mathcal{R}_m$ is also a σ-algebra and thus the set

$$G_1 = \bigcup \{F_r : r > 0, \quad r \quad rational\}$$

is measurable. Analogously the set $B = \{x : |g(x)| > |f(x)|\}$ is measurable and so $G_2 = B^c$ is measurable. Also $G_3 = G_2 \setminus G_1$ is measurable. Finally the function $h = f - \lambda$ is measurable, by Theorem 3.5, and so the set

$$G_4 = \bigcap_{n=1}^{\infty} \{x : |h(x)| < 1/n\}$$

is measurable.

Theorem 3.9 *Let $m \in M(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$ be measurable. Then f is $\tau_{\mathcal{R}_m}$-continuous at every point of the set*

$$Z = \bigcup_{p \in cs(E)} \{x : N_{m,p}(x) \neq 0\}.$$

Proof: Let $N_{m,p}(x) = d > 0$ and let $\epsilon > 0$. The set $G = \{x : |f(y) - f(x)| \leq \epsilon\}$ is measurable. Hence, there are $V, W \in \mathcal{R}$ such that $V \subset G \subset W$ and $m_p(W \setminus V) < d$. Since $x \in W$ and $N_{m,p}(x) > m_p(W \setminus V)$, it follows that $x \in V \subset G$, which proves that f is continuous at x.

Corollary 3.10 *Let $m \in M_\tau(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$ be measurable. If there exists an integrable function g such that $|f| \leq |g|$, then f is integrable.*

Proof : Given $p \in cs(E)$, there exists $W \in \mathcal{R}$ such that $\|g\|_W < \infty$ and $m_p(W^c) = 0$. By the preceding Theorem and the Theorem 2.9, f is $\bar{m}$-integrable and so f is m-integrable.

Theorem 3.11 *Let $m \in M(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let (f_n) be a sequence of measurable functions which converges to some f m-almost everywhere. Then f is measurable.*

Proof : Let A be a clopen subset of $\mathbb{K}$ and let $B_n = f_n^{-1}(A)$. The set $B = \liminf B_n$ is in $\mathcal{R}_m$ since $\mathcal{R}_m$ since is a σ-algebra. Let $Z = \{x : f(x) = \lim f_n(x)\}$. Then Z^c is m-negligible and hence measurable. Moreover, $f^{-1}(A) \cap Z = B \cap Z$. Indeed, let $x \in f^{-1}(A) \cap Z$. Since $\lim f_n(x) = f(x) \in A$, there exists a k such that
$x \in \bigcap_{n \geq k} B_n \subset B$. Conversely, if $x \in B \cap Z$, then there exists a k such that $x \in \bigcap_{n \geq k} B_n$, and so $f_n(x) \in A$ for all $n \geq k$. Since A is closed and $f_n(x) \to f(x)$, it follows that $f(x) \in A$ and so $x \in f^{-1}(A) \cap Z$. Now $B \cap Z$ is measurable and

$$f^{-1}(A) = [B \cap Z] \bigcup [f^{-1}(A) \cap Z^c].$$

As $f^{-1}(A) \cap Z^c$ is negligible, it is measurable and so $f^{-1}(A)$ is measurable. Hence the result follows.

Theorem 3.12 (**Egoroff's Theorem**) *Let $m \in M_\tau(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let (f_n) be a sequence of measurable functions which converges m-a.e to some f. Then for each $\epsilon > 0$ and each $p \in cs(E)$, there exists $A \in \mathcal{R}$, with $m_p(A^c) < \epsilon$, such that $f_n \to f$ uniformly on A.*

Proof : Let G be an m-negligible set such that $f_n(x) \to f(x)$ for all $x \in G^c$ and let $p \in cs(E)$ and $\epsilon > 0$. By the preceding Theorem, f is measurable.
Claim. For each $\delta > 0$, there exist $B \in \mathcal{R}$, with $m_p(B^c) \leq \epsilon$, and an integer N such that $|f_n(x) - f(x)| < \delta$ for all $x \in B$ and all $n \geq N$. In fact, let

$$A_n = \{x \in X : |f_n(x) - f(x)| \geq \delta\} \bigcap G^c \quad \text{and} \quad D_N = \bigcup_{n \geq N} A_n.$$

Since m is τ-additive, each $f_n - f$ is measurable (by Theorem 3.4) and so A_n is measurable, which implies that D_N is measurable since $\mathcal{R}$ is a σ-algebra. Moreover $D_N \downarrow \emptyset$ since $f_n(x) \to f(x)$ for all $x \in G^c$. As $\bar{m}$ is σ-additive, there exists an N such that $\bar{m}_p(D_N \cup G) = \bar{m}_p(D_N) < \epsilon$. There are $V, W \in \mathcal{R}$ such that $V \subset D_N \cup G \subset W$ and $m_p(W \setminus V) < \epsilon$. Now

$$m_p(W) = \max\{m_p(V), m_p(W \setminus V)\} \leq \max\{\bar{m}_p(D_N \cup G), \epsilon\} = \epsilon.$$

Taking $B = W^c$, we see that if $x \in B$, then $x \notin D_N \cup G$ and so $x \notin A_n$, for each $n \geq N$, i.e $|f_n(x) - f(x)| < \delta$. Thus the claim follows.
By our claim, there are $n_1 < n_2 < \ldots$, and sets $B_k \in \mathcal{R}$, with $m_p(B_k) < \epsilon$ and $|f_n - f(x)| < 1/k$ for all $x \notin B_k$ and all $n \geq n_k$. For $A = \bigcup B_k$, we have that $m_p(A) = \sup_k m_p(B_k) \leq \epsilon$. Moreover, $f_n \to f$ uniformly on A^c. In fact, given $\delta > 0$, choose $k > 1/\delta$. If $x \in A^c \subset B_k^c$, we have $|f_n(x) - f(x)| \leq 1/k < \delta$ for all $n \geq n_k$. This completes the proof.

Theorem 3.13 *Let $m \in M(\mathcal{R}, E)$, where E is metrizable, and let (f_n) be a sequence in $\mathbb{K}^X$ and $f \in \mathbb{K}^X$. If, for each $p \in cs(E)$ and each $\epsilon > 0$, there exists an A in $\mathcal{R}$, with $m_p(A) < \epsilon$, such that (f_n) converges uniformly to f on A^c, then $f_n(x) \to f(x)$ m-a.e.*

Proof : Let (p_n) be an increasing sequence of continuous seminorms on E such that, for each $p \in cs(E)$, there exists an n with $p \leq p_n$. For each k, there exists $A_k \in \mathcal{R}$, with $m_{p_k}(A_k) < 1/k$, such that $f_n \to f$ uniformly on A_k^c. Let $A = \bigcap A_k$ and let $p \in cs(E)$. Choose k such that $p \leq p_k$. Then, for each $n \geq k$, we have

$$m_p^*(A) \leq m_p(A_n) \leq m_{p_n}(A_n) < 1/n \to 0,$$

and hence A is negligible. Moreover, $f_n(x) \to f(x)$ for all $x \in A^c$.

4 Convergence in Measure

Let $m \in M(\mathcal{R}, E)$.

Definition 4.1 *A net (g_δ) in $\mathbb{K}^X$ converges in measure, with respect to m, to some $f \in \mathbb{K}^X$ if, for each $p \in cs(E)$ and each $\alpha > 0$, we have*

$$\lim_\delta m_p^*(\{x : |g_\delta(x) - f(x)| \geq \alpha\}) = 0.$$

Theorem 4.2 *Let $m \in M_\sigma(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let (f_n) be a sequence in $\mathbb{K}^X$ which converges in measure to both f and g. Then $f = g$ m-a.e.*

Proof : For each positive integer k, let

$$A_{nk} = \{x : |f_n(x) - f(x)| \geq 1/k\}, \quad B_{nk} = \{x : |g(x) - f_n(x)| \geq 1/k\},$$

$$G_k = \{x : |f(x) - g(x)| \geq 1/k\}.$$

Then $G_k \subset A_{nk} \bigcup B_{nk}$ and so

$$m_p^*(G_k) \leq \max\{m_p^*(A_{nk}), \quad m_p^*(B_{nk})\},$$

for all n. It follows that $m_p^*(G_k) = 0$ for all $p \in cs(E)$, and so G_k is negligible. Since m is σ-additive and $\mathcal{R}$ a σ-algebra, it follows that the set

$$G = \{x : f(x) \neq g(x)\} = \bigcup G_k$$

is negligible, and thus $f = g$ m-a.e

Theorem 4.3 *Let $m \in M(\mathcal{R}, E)$ and $f \in \mathbb{K}^X$. Then, f is m-integrable iff*

1. *There exists a net (g_δ) in $S(\mathcal{R})$ which converges in measure to f.*
2. *For each $p \in cs(E)$ there exists a $W \in \mathcal{R}$, with $m_p(W^c) = 0$, such that f is bounded on W.*

Proof : Assume that f is integrable. Then (2) holds by Theorem 2.1. To prove (1), we consider the set $\Delta = \{(n,p) : n \in \mathbb{N}, p \in cs(E)\}$. We make Δ into a directed set by defining $(n_1, p_1 \geq (n_2, p_2)$ iff $n_1 \geq n_2$ and $p_1 \geq p_2$.
Claim: For each $\delta = (n,p)$, there exist $h_\delta \in S(\mathcal{R})$ and $G_\delta \in \mathcal{R}$ such that

$$m_p(G_\delta) < 1/n \quad \textit{and} \quad A_\delta = \{x : |h_\delta(x) - f(x)| \geq 1/n\} \subset G_\delta.$$

Moreover, we can choose h_δ so that $h_\delta(X) \subset f(X)$.
Indeed, there exists an $\mathcal{R}$-partition $\{B_1, \dots, B_N\}$ of X such that , for each $1 \leq k \leq N$, we have $|f(x) - f(y)| \cdot m_p(B_k) < 1/n^2$ if $x, y \in B_k$. Choose $x_k \in B_k$ and set $g_\delta = \sum_{k=1}^{N} f(x_k)\chi_{B_k}$. Let

$$A_\delta = \{x : |h_\delta(x) - f(x)| \geq 1/n\} \quad \textit{and} \quad G_\delta = \bigcup\{B_k : B_k \cap A_\delta \neq \emptyset\}.$$

If $x \in B_k \cap A_\delta$, then

$$1/n^2 > |f(x) - f(x_k)| \cdot m_p(B_k) \geq 1/n \cdot m_p(B_k),$$

and so $m_p(B_k) < 1/n$. It follows that $m_p(G_\delta) < 1/n$ and clearly $A_\delta \subset G_\delta$. This proves the claim. Now $h_\delta \to f$ in measure. In fact, let $p_o \in cs(E)$, $\alpha > 0$ and $\epsilon > 0$. Choose $n_o > 1/\alpha, 1/\epsilon$. For $\delta = (n,p) \geq \delta_o = (n_o, p_o)$, let

$$Z_\delta = \{x : |g_\delta(x) - f(x)| \geq \alpha\}.$$

Then $Z_\delta \subset A_\delta \subset G_\delta$ and so $m_p^*(Z_\delta) \leq m_p(G_\delta) < 1/n < \epsilon$. This proves that $h_\delta \to f$ in measure.
Conversely, suppose that (1) and (2) hold and let $p \in cs(E)$ and $\epsilon > 0$. By (2), there exists $W \in \mathcal{R}$, with $m_p(W^c) = 0$, such that $\|f\|_W < d < \infty$. Let (g_δ) be a net in $S(\mathcal{R})$ which converges in measure to f. Choose $\alpha > 0$ such that $\alpha \cdot m_p(X) < \epsilon$. There exists a δ_o such that $m_p^*(Z_{\delta_o}) < \epsilon/d$, where

$$Z_{\delta_o} = \{x : |g_{\delta_o}(x) - f(x)| \geq \alpha\}.$$

There exist an $\mathcal{R}$-partition $\{W_1, \dots, W_N\}$ of X and $\lambda_i \in \mathbb{K}$ such that $g_{\delta_o} = \sum_{i=1}^{N} \lambda_i \chi_{W_i}$. There is a $V \in \mathcal{R}$ containing Z_{δ_o} such that $m_p(V) < \epsilon/d$. Let $\{V_1, \dots, V_n\}$ be any $\mathcal{R}$-partition of X, which is a refinement of each of the partitions $\{W_1, \dots, W_N\}$, $\{W, W^c\}$, and $\{V, V^c\}$. Let $1 \leq i \leq n$ and $x, y \in V_i$. We will prove that

$$|f(x) - f(y)| \cdot m_p(V_i) \leq \epsilon.$$

This is clearly true if $V_i \subset W^c$. So, assume that $V_i \subset W$. If $V_i \subset V$, then

$$|f(x) - f(y)| \cdot m_p(V_i) \leq d \cdot m_p(V) \leq \epsilon.$$

Finally, if $V_i \subset V^c$, then (since $g_{\delta_o}(x) = g_{\delta_o}(y)$ as x, y are in some W_j) we have

$$|f(x) - f(y)| \leq \max\{|f(x) - g_{\delta_o}(x)|, \quad |g_{\delta_o}(y) - f(y)|\} < \alpha$$

and so

$$|f(x) - f(y)| \cdot m_p(V_i) \leq \alpha \cdot m_p(X) < \epsilon.$$

Now the result follows from Theorem 2.1.

Theorem 4.4 *Let $m \in M(\mathcal{R}, E)$ and let $(g_\delta)_{\delta \in \Delta}$ be a net in $\mathbb{K}^X$ which converges in measure to some f. If E is metrizable, then there exist $\delta_1 \leq \delta_2 \leq \ldots$ such that the sequence (g_{δ_n}) converges in measure to f.*

Proof : Let (p_n) be an increasing sequence of continuous seminorms on E such that, for each $p \in cs(E)$, there exists an n with $p \leq p_n$. There is an increasing sequence (δ_n) in Δ such that

$$m^*_{p_n}(\{x : |g_\delta(x) - f(x)| \geq 1/n\}) < 1/n$$

for all $\delta \geq \delta_n$. Let $h_n = g_{\delta_n}$. Then $h_n \to f$ in measure. Indeed, let $p \in cs(E)$, $\alpha > 0$ and $\epsilon > 0$. Choose $n_o > 1/\alpha, 1/\epsilon$ with $p_{n_o} \geq p$. Then, for $n \geq n_o$, we have

$$\begin{aligned} m^*_p(\{x : |h_n(x) - f(x)| \geq \alpha\}) &\leq m^*_p(\{x : |h_n(x) - f(x)| \geq 1/n\}) \\ &\leq m^*_{p_n}(\{x : |h_n(x) - f(x)| \geq 1/n\}) < 1/n < \epsilon. \end{aligned}$$

Thus $h_n \to f$ in measure and the result follows.

Corollary 4.5 *If $f \in \mathbb{K}^X$ is m-integrable and E metrizable, then there exists a sequence (g_n) in $S(\mathcal{R})$ which converges in measure to f. Moreover, we can choose (g_n) so that $g_n(X) \subset f(X)$ for all n.*

Theorem 4.6 *Let $m \in M_\sigma(\mathcal{R}, E)$, where E is metrizable, and consider on X the topology $\tau_{\mathcal{R}}$. Let (f_n) be a sequence in $\mathbb{K}^X$ which converges in measure to some f. Then, there exist a subsequence (f_{n_k}) and an F_σ set F such that F is a support set for m and $f_{n_k} \to f$ pointwise on F. If $\mathcal{R}$ is a σ-algebra, then we may choose F to be in $\mathcal{R}$.*

Proof : Let (p_n) be an increasing sequence of continuous seminorms on E such that, for each $p \in cs(E)$, there exists an n with $p \leq p_n$. Choose inductively $n_1 < n_2 < \ldots$ such that

$$m^*_{p_k}(\{x : |f_n(x) - f(x)| \geq 1/k\}) < 1/k$$

for all $n \geq n_k$. Let

$$A_k = \{x : |f_n(x) - f(x)| \geq 1/k\}$$

and let $V_k \in \mathcal{R}$, containing A_k, such that $m_{p_k}(V_k) < 1/k$. Set

$$A = \bigcap_{N=1}^{\infty} \bigcup_{k \geq N} V_k, \quad F = X \setminus A.$$

Then F is an F_σ set and $F \in \mathcal{R}$ if $\mathcal{R}$ is a σ-algebra. If $V \in \mathcal{R}$ is contained in A, then $p_k(m(V)) = 0$ for all k. Indeed, for all N, we have $V \subset \bigcup_{i \geq N} V_i$. So, if $N > k$, then

$$m_{p_k}(V) \leq \sup_{i \geq N} m_{p_k}(V_i) \leq \sup_{i \geq N} m_{p_i}(V_i) \leq 1/N$$

and so $m_{p_k}(V) = 0$. This proves that F is a support set for m. Finally, let $x \in F$ and let N_o be such that $x \notin \bigcup_{i \geq N_o} V_i$. For $k \geq N_o$, we have $x \notin V_k$ and so $|f_{n_k}(x) - f(x)| < 1/k \to 0$. This clearly completes the proof.

Theorem 4.7 *Let $m \in M_\sigma(\mathcal{R}, E)$, where E is metrizable and $\mathcal{R}$ a σ-algebra. If f is m-integrable, then f is m-measurable.*

Proof : By Corollary 4.5, there exists a sequence (g_n) in $S(\mathcal{R})$ which converges in measure to f. In view of the preceding Theorem, there exist a subsequence (g_{n_k}) and a set $F \in \mathcal{R}$ such that F is a support set for m and $g_{n_k} \to f$ pointwise on F. Since each g_{n_k} is measurable, it follows that f is measurable by Theorem 3.11.

Theorem 4.8 *Let $m \in M_\sigma(\mathcal{R}, E)$, where E is metrizable and $\mathcal{R}$ a σ-algebra. If a sequence (f_n) of measurable functions converges in measure to some f, then f is measurable.*

Proof : By Theorem 4.6 there exist a se subsequence (f_{n_k}) and a set $F \in \mathcal{R}$ such that F is a support set for m and $f_{n_k} \to f$ pointwise on F. Now the result follows from Theorem 3.11.

Theorem 4.9 *Let $m \in M_\sigma(\mathcal{R}, E)$, $p \in cs(E)$ and $\epsilon > 0$. Then :*

1. *If $f \in \mathbb{K}^X$ is measurable, then there exists a $d > 0$ such that*

$$m_p^*(\{x : |f(x)| > d\}) < \epsilon.$$

2. *If (g_n) is a sequence of measurable functions which converges in measure to some g, then there exists $\alpha > 0$ such that $m_p^*(\{x : |g(x)| > \alpha\}) < \epsilon$.*

Proof : 1). Let $V_n = \{x : |f(x)| > n\}$. Then $V_n \in \mathcal{R}_m$ and $V_n \downarrow \emptyset$. Since $\bar{m}$ is σ-additive, there exists an n such that $\bar{m}_p^*(V_n) < \epsilon$.
2). Let $A_n = \{x : |g_n(x) - g(x)| \geq 1\}$. There exists an n such that $m_p^*(A_n) < \epsilon$. By (1), there exists $\alpha > 1$ sauch that, if $B = \{x : |g_n(x)| > \alpha\}$, then $m_p^*(B) < \epsilon$. If $A = \{x : |g(x)| > \alpha\}$, then $A \subset B \cup A_n$ and so

$$m_p^*(A) \leq \max\{m_p^*(B), m_p^*(A_n)\} < \epsilon.$$

Theorem 4.10 *Let $m \in M_\sigma(\mathcal{R}, E)$ and let (f_n) and (g_n) be two sequences of measurable functions which converge in measure to f, g, respectively. Then $f_n + g_n \to f + g$ and $f_n g_n \to fg$ in measure.*

Proof : It is easy to see that $(f_n + g_n)$ converges in measure to $f + g$. To prove that the sequence $(f_n g_n)$ converges in measure to fg, we first prove that $f_n g \to fg$ in measure. Indeed, let $p \in cs(E)$, $\alpha > 0$ and $\epsilon > 0$. By the preceding Theorem, there exists a $d > 0$ such that, if $A = \{x : |g(x)| > d\}$, then $m_p^*(A) < \epsilon$. Let

$$A_n = \{x : |f_n(x)g(x)(-f(x)g(x)| \geq \alpha\}, \quad B_n = \{x : |f_n(x) - f(x)| \geq \alpha/d\}.$$

Then $A_n \subset B_n \cup A$. There exists an n_o such that $m_p^*(B_n) < \epsilon$ for $n \geq n_o$. Thus, for $n \geq n_o$, we have

$$m_p^*(A_n) \leq \max\{m_p^*(B_n), m_p^*(A)\} < \epsilon,$$

which proves our claim.
Next we show that $f_n^2 \to f^2$ (and analogously $g_n^2 \to g^2$) in measure. Indeed let $h_n = f_n - f$. Then $h_n \to 0$ in measure. Since, for $\alpha > 0$, we have

$$\{x : |h_n^2(x)| \geq \alpha\} = \{x : |h_n(x)| \geq \alpha^{1/2}\},$$

it follows that $h_n^2 \to 0$ in measure. Now $f_n^2 - f^2 = h_n^2 + 2(f_n f - f^2) \to 0$ in measure and so $f_n^2 \to f^2$ in measure.
Next we observe that

$$(f_n + g_n)(f + g) = f_n f + g_n f + f_n g + g_n g \to f^2 + 2fg + g^2$$

in measure. If $\phi_n = (f_n + g_n) - (f + g)$, then $\phi_n \to 0$ in measure and so $\phi_n^2 \to 0$ in measure. Now

$$(f_n + g_n)^2 - (f + g)^2 = \phi_n^2 + 2\left[(f_n + g_n)(f + g) - (f + g)^2\right] \to 0$$

in measure. Finally,

$$f_n g_n = \frac{1}{2}\left[(f_n + g_n)^2 - f_n^2 - g_n^2\right] \to \frac{1}{2}\left[(f + g)^2 - f^2 - g^2\right] = fg$$

in measure. Hence the result follows.

Theorem 4.11 *Let $m \in M_\sigma(\mathcal{R}, E)$, where E is metrizable and $\mathcal{R}$ a σ-algebra. Let $f, g \in \mathbb{K}^X$ be such that f is m-integrable and g m-measurable. Then $f + g$ and gf are m-measurable.*

Proof : By Corollary 4.5, there exists a sequence (h_n) of $\mathcal{R}$-simple functions which converges in measure to f. In view of the preceding Theorem, the sequence $(h_n g)$ converges in measure to fg. Each $h_n g$ is measurable by Theorem 3.5. Hence fg is measurable by Theorem 4.8. The same Theorem implies that $f + g$ is measurable since $h_n + g \to f + g$ in measure and each $h_n + g$ is measurable by Theorem 3.5.

Theorem 4.12 *Let $m \in M_\tau(\mathcal{R}, E)$ and let $(f_\delta)_{\delta \in \Delta}$ be a net in $\mathbb{K}^X$ which converges in measure to some f. Then, there exists a support set F for m and a subnet of (f_δ) which converges to f pointwise on F.*

Proof : Let $\Xi = \{(\delta, p, k) : \delta \in \Delta, p \in cs(E), k \in \mathbb{N}\}$ and make Ξ into a directed set by defining $(\delta, p, k) \geq (\delta_1, p_1, k_1)$ iff $\delta \geq \delta_1$, $p \geq p_1$ and $k \geq k_1$. Let $\xi = (\delta, p, k)$. There exists $\delta_1 = \psi(\xi) \geq \delta$ such that

$$m_p^*(\{x : |f_{\delta_1}(x) - f(x)| \geq 1/k\}) < 1/k.$$

In this way we get a subnet $(f_{\psi(\xi)})_{\xi \in \Xi}$ of (f_δ). Let

$$G_\xi = \{x : |f_{\psi(\xi)}(x) - f(x)| \geq 1/k\}$$

and choose $W_\xi \in \mathcal{R}$ containing G_ξ and such that $m_p(W_\xi) < 1/k$. Let

$$A = \bigcap_{\xi \in \Xi} \bigcup_{\xi' \geq \xi} W_{\xi'}, \quad F = X \setminus A.$$

Then : 1. $f_{\psi(\xi)}(x) \to f(x)$ for all $x \in F$. In fact, let $x \in F$. There exists a $\xi_1 = (\delta_1, p_1, k_1)$ such that Now, for $\xi = (\delta, p, k) \geq \xi_1$, we have

$$|f_{\psi(\xi)}(x) - f(x)| < 1/k \to 0 \quad as \quad k \to \infty.$$

Thus $f_{\psi(\xi)}(x) \to f(x)$.
2. F is a support set for m. Indeed, Let $W \in \mathcal{R}$ be contained in A and let $\xi_o = (\delta_o, p_o, k_o) \in \Xi$. Then $W \subset \bigcup_{\xi' \geq \xi_o} W_{\xi'}$. Since m is τ-additive, we have

$$m_{p_o}(W) \leq \sup_{\xi' \geq \xi_o} m_{p_o}(W_{\xi'}).$$

But, for $\xi' = (\delta, p, k) \geq \xi_o$, we have

$$m_{p_o}(W_{\xi'}) \leq m_p(W_{\xi'}) < 1/k \leq 1/k_o.$$

It follows that $m_{p_o}(W) = 0$ for all $p_o \in cs(E)$, which proves that F is a support set for m. This completes the proof.

Theorem 4.13 (**Dominated Convergence Theorem**) *Let $m \in M_\tau(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra and E metrizable, and let (f_n) be a sequence of integrable functions which converges m-a.e to some f. If there exists an integrable function g such that $|f_n| \leq |g|$ for all n, then f is integrable and*

$$\int f\, dm = \lim \int f_n\, dm.$$

Proof : Let $p \in cs(E)$ and $\epsilon > 0$. Since g is integrable, there exists a $W \in \mathcal{R}$ such that $m_p(W^c) = 0$ and $\|g\|_W < d < \infty$. Each f_n is measurable by Theorem 4.7. By Egoroff's Theorem, there exists $A \in \mathcal{R}$, with $m_p(A^c) < \epsilon/d$, such that $f_n \to f$ uniformly on A. Also, there exists an m-negligible set B such that $f_n(x) \to f(x)$ for all $x \in B^c$. Clearly $|f| \leq |g|$ on B^c. For each k, there exists

$B_k \in \mathcal{R}$ with $B \subset B_k$ and $m_p(B_k) < 1/k$. The set $F = \bigcap B_k$ is in $\mathcal{R}$ and $m_p(F) = 0$. Since $f_n \to f$ uniformly on A, there exists n_o such that

$$\|f_n - f\|_A < \min\{\epsilon/d, \epsilon/\|m\|_p\}.$$

for all $n \geq n_o$. Let now $n \geq n_o$. Since f_n is integrable, there exists an $\mathcal{R}$-partition $\{A_1, \ldots, A_N\}$ of X, which is a refinement of each of the partitions $\{F, F^c\}$, $\{W, W^c\}$, $\{A, A^c\}$, such that, for all $1 \leq k \leq N$, we have $|f_n(x) - f_n(y)| \cdot m_p(A_k) < \epsilon$ if $x, y \in A_k$. Now, if $x, y \in A_k$, then $|f(x) - f(y)| \cdot m_p(A_k) \leq \epsilon$. In fact, this is clearly true if $A_k \subset W^c$ or $A_k \subset F$. So assume that $A_k \subset F^c \cap W$. Then, for $x, y \in A_k$, we have

$$|f(x) - f(y)) \leq \max\{|f(x) - f_n(x)|, |f_n(x) - f_n(y)|, |f_n(y) - f(y)|\}.$$

It follows from this that $|f(x) - f(y)| \cdot m_p(A_k) \leq \epsilon$. This proves that f is m-integrable. Moreover, if $x_k \in A_k$, then

$$p\left(\int f\,dm - \sum_{k=1}^{N} f(x_k)m(A_k)\right), \quad p\left(\int f_n\,dm - \sum_{k=1}^{N} f_n(x_k)m(A_k)\right) \leq \epsilon.$$

Also, for $1 \leq k \leq N$, we have $|f(x_k) - f_n(x_k)| \cdot p(m(A_k) \leq \epsilon$. Indeed, this is clearly true if $A_K \subset W^c$ or $A_k \subset F$. So assume that $A_k \subset F^c \cap W$. If $A_k \subset A$, then

$$|f(x_k) - f_n(x_k)| \cdot p(m(A_k) \leq \|f - f_n\|_A \cdot \|m\|_p \leq \epsilon,$$

while for $A_k \subset A^c$, we have

$$|f(x_k) - f_n(x_k)| \cdot p(m(A_k \leq d \cdot m_p(A^c) \leq \epsilon.$$

It follows from the above that

$$p\left(\int f\,dm - \int f_n\,dm\right) \leq \epsilon$$

for alll $n \geq n_o$. Thus

$$\int f\,dm = \lim \int f_n\,dm.$$

Theorem 4.14 *Let $m \in M_\tau(\mathcal{R}, E)$, where E is metrizable and $\mathcal{R}$ a σ-algebra, and let $f \in \mathbb{K}^X$. Then, f is m-integrable iff it is measurable (equivalently $\tau_{\mathcal{R}_m}$-continuous) and, for each $p \in cs(E)$, there exists a $W \in \mathcal{R}$ such that $m_p(W^c) = 0$ and f is bounded on W.*

Proof : The necessity follows from Theorems 4.7 and 2.1. Conversely, suppose that the condition is satisfied. We will show that f is $\bar{m}$-integrable and hence m-integrable. Let $p \in cs(E)$, $\epsilon > 0$ and let $W \in \mathcal{R}$ be such that f is bounded on W and $m_p(W^c) = 0$. Let $f_1 = f \cdot \chi_W$. Since f is measurable, it is $\tau_{\mathcal{R}_m}$-continuous (by theorem 3.6) and so f_1 is $\bar{m}$-integrable by [7, Theorem 4.11]. Hence there exists a $\mathcal{R}_m$-partition $\{A_1, \ldots, A_n\}$ of X such that, for all $1 \leq k \leq n$, we have $|f_1(x) - f_1(y)| \cdot m_p(A_k) < \epsilon$ if $x, y \in A_k$. Let now $\{B_1, \ldots, B_N\}$ be any $\mathcal{R}_m$-partition of X which is a refinement of both $\{A_1, \ldots, A_n\}$ and $\{W, W^c\}$. Then, for $1 \leq k \leq N$ and $x, y \in B_k$, we have $|f(x) - f(y)| \cdot m_p(B_k) < \epsilon$. Indeed, this clearly holds if $B_k \subset W^c$. Suppose that $B_k \subset W$. Then $f = f_1$ on B_k and so $|f(x) - f(y)| \cdot m_p(B_k) < \epsilon$ since B_k is contained in some A_i. Now the result follows.

Theorem 4.15 *Let $m \in M_\tau(\mathcal{R}, E)$, where $\mathcal{R}$ is a σ-algebra, and let (f_n) be a sequence of measurable functions which converges m-a.e. to some f. Then $f_n \to f$ in measure and f is measurable,*

Proof : Let $p \in cs(E)$, $\alpha > 0$ and $A_n = \{x : |f_n(x) - f(x)| \geq \alpha\}$. Given $\epsilon > 0$, there exists (by Egoroff's Theorem) a set $A \in \mathcal{R}$, with $m_p(A^c) < \epsilon$, such that $f_n \to f$ uniformly on A. Hence, there exists an n_o such that $\|f_n - f\|_A < \alpha$ for all $n \geq n_o$. Now, for $n \geq n_o$, we have $A_n \subset A^c$ and so $m_p^*(A_n) \leq m_p(A^c) < \epsilon$. Hence $f_n \to f$ in measure. Also f is measurable by Theorem 3.11.

Theorem 4.16 *$m \in M_\tau(\mathcal{R}, E)$ and let $f \in \mathbb{K}^X$ be measurable. Then, there exists a net (g_δ) in $S(\mathcal{R})$ which converges in measure to f. In case E is metrizable, there exists a sequence (h_n) in $S(\mathcal{R})$ converging to f in measure.*

Proof : We prove first the following
Claim : For each $\epsilon > 0$ and each $p \in cs(E)$, there exist $A \in \mathcal{R}$, with $m_p(A^c) < \epsilon$, and $g \in S(\mathcal{R})$ such that $\|f - g\|_A \leq \epsilon$. In fact, consider the equivalence relation $\sim$ on X, $x \sim y$ iff $|f(x) - f(y)| \leq \epsilon$. Let $(B_i)_{i \in I}$ be the family of all equivalence classes and let $x_i \in B_i$. Then $B_i = \{x : |f(x) - f(x_i)| \leq \epsilon\}$ and so B_i is measurable since f is measurable. For $J \subset I$ finite, let $G_J = \left(\bigcup_{i \in J} B_i\right)^c$. Then G_J is measurable and $G_J \downarrow \emptyset$. Since $\bar{m}$ is τ-additive, there exists a $J = \{i_1, \ldots, i_n\}$ such that $\bar{m}_p(G_J) < \epsilon$. For $1 \leq r \leq n$, there are $V_r, W_r \in \mathcal{R}$ such that $V_r \subset B_{i_r} \subset W_r$ and $m_p(W_r \setminus V_r) < \epsilon$. Let $y_r \in V_r$ and $g = \sum_{r=1}^n f(y_r)\chi_{V_r}$. If $A = \bigcup_{r=1}^n V_r$, then

$$A^c = \bigcap_{r=1}^n V_r^c \subset G_J \bigcup \left(\bigcup_{r=1}^n W_r \setminus V_r \right).$$

Thus,

$$m_p(A^c) = \bar{m}_p(A^c) \leq \max\{\bar{m}_p(G_J), m_p(W_1 \setminus V_1), \ldots, m_p(W_n \setminus V_n)\} < \epsilon.$$

Moreover, if $x \in A$, then $x \in V_r$, for some r, and so $|f(x) - g(x)| = |f(x) - f(y_r)| \leq \epsilon$. thus $\|f - g\|_A \leq \epsilon$ and the claim follows.
Let now $\Delta = \{(n, p) : n \in \mathbb{N}, p \in cs(E)\}$. For $\delta = (n, p) \in \Delta$, there exist a

function $g_\delta \in S(\mathcal{R})$ and a set $G_\delta \in \mathcal{R}$ such that $m_p(G_\delta^c) < 1/n$ and $\|g - g_\delta\|_{G_\delta} < 1/n$. Then $g_\delta \to f$ in measure. Indeed, let $p_o \in cs(E)$ and $\alpha, \epsilon > 0$. Choose $n_o > 1/\alpha, 1/\epsilon$ and set $\delta_o = (n_o, p_o)$. If $\delta = (n, p) \geq \delta_o$, then

$$\begin{aligned} m_{p_o}^*(\{x : |g_\delta(x) - f(x)| \geq \alpha\} &\leq \bar{m}_p(\{x : |g_\delta(x) - f(x)| \geq \alpha\} \\ &\leq \bar{m}_p(\{x : |g_\delta(x) - f(x)| \geq 1/n\} \\ &\leq m_p(G_\delta^c) < 1/n < \epsilon. \end{aligned}$$

This proves that $g_\delta \to f$ in measure. The last part of the Theorem follows from Theorem 4.4.

Corollary 4.17 *Let $m \in M_\tau(\mathcal{R}, E)$, where E is metrizable and $\mathcal{R}$ a σ-algebra, and let $f \in \mathbb{K}^X$. Then f is measurable iff there exists a sequence (h_n) in $S(\mathcal{R})$ converging in measure to f.*

Proof : The necessity follows from the preceding Theorem. Conversely let (h_n) in $S(\mathcal{R})$ converging in measure to f. By Theorem 4.6, there exist a subsequence (h_{n_k}) and $F \in \mathcal{R}$ such that F is a support set for m and $h_{n_k} \to f$ pointwise on F. Hence f is measurable by Theorem 3.11.

Theorem 4.18 (**Lusin's Theorem**) *Let $m \in M_\tau(\mathcal{R}, E)$, where E is metrizable and $\mathcal{R}$ a σ-algebra, and let $f \in \mathbb{K}^X$. Then f is measurable iff, for each $p \in cs(E)$ and each $\epsilon > 0$, there exist $A \in \mathcal{R}$, with $m_p(A^c) < \epsilon$, and a $\tau_\mathcal{R}$-continuous function g such that $f(x) = g(x)$ for all $x \in A$.*

Proof : Suppose that f is measurable and let $p \in cs(E)$, $\epsilon > 0$. By the preceding Corollary, there exists a sequence (h_n) in $S(\mathcal{R})$ which converges in measure to f. Each h_n is measurable. By theorem 4.6 there exist a subsequence $(g_k) = (h_{n_k})$ and $F \in \mathcal{R}$ such that F is a support set for m and $g_k \to f$ pointwise on F. By Egoroff's Theorem, there exists $A \in \mathcal{R}$, with $m_p(A^c) < \epsilon$, such that $g_k \to f$ uniformly on A. Since A is $\tau_\mathcal{R}$-open and each g_k is $\tau_\mathcal{R}$-continuous, it follows that f is $\tau_\mathcal{R}$-continuous at every point of A. If $g = \chi_A f$, then g is $\tau_\mathcal{R}$-continuous and $g = f$ on A. Conversely, suppose that the condition is satisfied and let B be a clopen subset of $\mathbb{K}$ and $p \in cs(E)$. We need to show that $f^{-1}(B) \in \mathcal{R}_m$. For each positive integer k, there exist $A_k \in \mathcal{R}$, with $m_p(A_k^c) < 1/k$, and a $\tau_\mathcal{R}$-continuous function u_k such that $u_k = f$ on A_k. Let

$$A = \bigcup_k A_k, \quad F = f^{-1}(B) \cap A, \quad G = f^{-1}(B) \cap A^c.$$

Then

$$F = \bigcup_{k=1}^{\infty} f^{-1}(B) \cap A_k = \bigcup_{k=1}^{\infty} u_k^{-1}(B) \cap A_k.$$

Since u_k is $\tau_{\mathcal{R}}$-continuous (and hence $\tau_{\mathcal{R}_m}$-continuous), it follows that u_k is m-measurable and so $F \in \mathcal{R}_m$. Moreover, $G \subset A_k^c$, for each k, and so

$$f^{-1}(B) \triangle F = G \subset A_k^c,$$

which implies that $d_p(f^{-1}(B), F) \leq m_p(A_k^c) < 1/k \to 0$. This proves that $f^{-1}(B)$ belongs to the closure of $\mathcal{R}_m$ in $P(X)$ and hence $f^{-1}(B) \in \mathcal{R}_m$. This completes the proof.

Definition 4.19 *Let $m \in M(\mathcal{R}, E)$. A sequence (f_n) in $\mathbb{K}^X$ is said to be Cauchy in measure if, for each $p \in cs(E)$ and each $\alpha > 0$, we have*

$$\lim_{n,r \to \infty} m_p^*(\{x : |f_n(x) - f_r(x)| \geq \alpha\}) = 0.$$

We have the following easily verified

Lemma 4.20 *If $f_n \to f$ in measure, then (f_n) is Cauchy in measure.*

Theorem 4.21 *Let $m \in M_\sigma(\mathcal{R}, E)$ and suppose that E is metrizablre and $\mathcal{R}$ a σ-algebra. If (f_n) is a sequence of measurable functions which is Cauchy in measure, then there exists an f such that $f_n \to f$ in measure.*

Proof : Let (p_n) be an increasing sequence of continuous seminorms on E such that, for each $p \in cs(E)$, the exists an n with $p \leq p_n$. There are $n_1 < n_2 < \ldots$ such that

$$m_{p_k}^*(\{x : |f_n(x) - f_r(x)| \geq 1/k\}) < 1/k$$

for all $n, r \geq n_k$. Let $h_k = f_{n_k}$ and let $A_k \in \mathcal{R}$ such that $m_{p_k}(A_k) < 1/k$ and

$$\{x : |h_k(x) - h_{k+1}(x)| \geq 1/k\} \subset A_k.$$

Let $F_k = \bigcup_{i \geq k} A_i$. Then $F_k \in \mathcal{R}$ and

$$m_{p_k}(F_k) = \sup_{i \geq k} m_{p_k}(A_i) \leq \sup_{i \geq k} m_{p_i}(A_i) \leq 1/k.$$

On each $X \setminus F_k$, the sequence (h_j) converges uniformly. In fact, let $\epsilon > 0$ and choose $n_o > k, 1/\epsilon$. If $i, j \geq n_o$, then for $x \notin F_k$ we have $|h_i(x) - h_j(x)| < 1/n_o < \epsilon$. It follows now that the $\lim h_j(x)$ exists for every $x \notin F = \bigcap F_k$. Define f on X by $f(x) \doteq \lim h_j(x)$ when $x \notin F$ and arbitrarily when $x \in F$. We will show that $f_n \to f$ in measure. Indeed, let $p \in cs(E)$, $\alpha > 0$ and $\epsilon > 0$. Set

$$B_n = \{x : |f_n(x) - f(x)| \geq \alpha\}.$$

Choose $r > 1/\epsilon$ such that $p \leq p_{n_r}$ and $n_r > 1/\alpha$. Since $h_j \to f$ uniformly on $F_{n_r}^c$, there exists $j \geq r, 1/\alpha$ such that $|h_j(x) - f(x)| < \alpha$ for all $x \in F_{n_r}^c$. Let now $n \geq n_j$. Then $B_n \subset G_1 \bigcup G_2$, where

$$G_1 = \{x : |f_n(x) - f_{n_j}(x)| \geq \alpha\}, \quad \text{and} \quad G_2 = \{x : |f_{n_j}(x) - f(x)| \geq \alpha\}.$$

Moreover $G_2 \subset F_{n_r}$ and so

$$m_p^*(G_2) \leq m_p(F_{n_r}) \leq m_{p_{n_r}}(F_{n_r}) < 1/r < \epsilon.$$

Also

$$G_1 \subset \{x : |f_n(x) - f_{n_j}(x)| \geq 1/j\}$$

and thus

$$m_p^*(G_1) \leq m_{p_{n_j}}(G_1) < 1/j < \epsilon.$$

Hence $m_p^*(B_n) < \epsilon$ for all $n \geq n_j$. This clearly completes the proof.

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